Worcester County Mathematics League

Varsity Meet 3
January 7, 2015

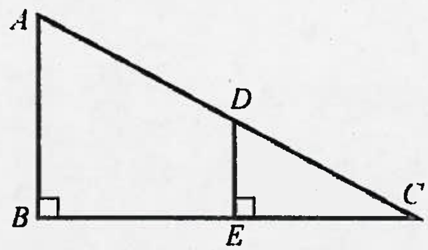
COACHES' COPY
ROUNDS, ANSWERS, AND SOLUTIONS

WORCESTER COUNTY MATHEMATICS LEAGUE

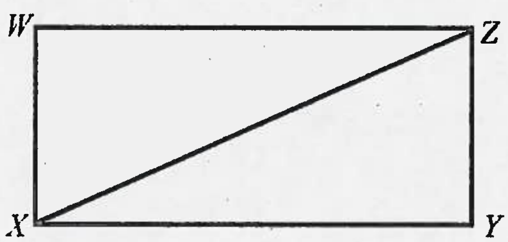
Varsity Meet 3 – January 7, 2015
Round 1: Similarity and Pythagorean Theorem

All answers must be in simplest exact form in the answer section
NO CALCULATOR ALLOWED

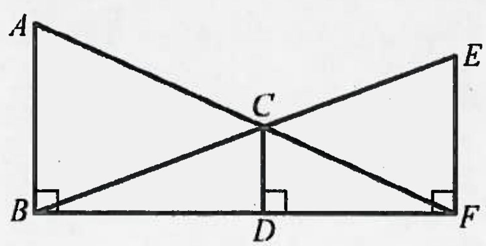
1. In the figure, ΔABC is similar to ΔDEC, $AB = 9$, $AC = 24$, and $DE = 3$. Find AD.



2. Rectangle $WXYZ$ has side lengths 5 and 12. What is the shortest distance from point W to diagonal $\overline{XZ}$?



3. In the figure, $AB = 50$ and $EF = 19\frac{4}{9}$. Find the length of $\overline{CD}$.



ANSWERS

(1 pt.) 1. ______________________

(2 pts.) 2. ______________________

(3 pts.) 3. ______________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 3 – January 7, 2015
Round 2: Algebra 1

All answers must be in simplest exact form in the answer section
NO CALCULATOR ALLOWED

1. In a mayoral election, the ratio of votes for Ann Alberts to votes for Barney Brooks was $6:5$. If a total of 4191 votes were cast, how many votes did Ann Alberts receive?

2. A 20 gallon radiator containing a solution of 30% antifreeze is to be partially drained, then refilled with 75% antifreeze to bring the resulting antifreeze concentration to 50%. How much of the original 30% solution must be drained?

3. During the first two fifths of the distance of his commute, Kevin's average speed is 20 miles per hour due to traffic. If his average speed for the entire commute is $32\frac{1}{7}$ miles per hour, what is his average speed during the last three fifths of the distance of his trip?

ANSWERS

(1 pt.) 1. ______________________ votes

(2 pts.) 2. ______________________ gallons

(3 pts.) 3. ______________________ miles per hour

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 3 – January 7, 2015
Round 3: Functions

All answers must be in simplest exact form in the answer section
NO CALCULATOR ALLOWED

1. If $f(x) = x^2 - x - 1$ and $g(x) = 3x - 7$, find the value of $g(f(-3))$.

2. If $f(x) = x + 4$ and $f(g(x)) = -2x - 5$, then $g(f(x))$ can be written as $ax + b$ for some real numbers a and b. Find the value of $a + b$.

3. An invertible function f with domain $(0, \infty)$ satisfies $f(ab) = f(a) + f(b)$ for all a and b in its domain. If $f^{-1}(81) = 8$, find $f^{-1}(10)$.

ANSWERS

(1 pt.) 1. ______________________

(2 pts.) 2. ______________________

(3 pts.) 3. ______________________

Varsity Meet 3 – January 7, 2015
Round 4: Combinatorics

All answers must be in simplest exact form in the answer section
NO CALCULATOR ALLOWED

1. For a club of 10 members, in how many different ways can the offices of president, secretary, and treasurer be filled?

2. A manufacturer of dress shoes has available 4 different dyes, 7 types of bows, and 5 types of straps. Each shoe must be dyed, but may or may not contain a bow or a strap. How many different styles of shoe can the manufacturer make?

3. How many distinct 5–letter "words" (ordered strings of letters, which need not be English words) can be formed from the letters in the word:

SNOWMAN

ANSWERS

(1 pt.) 1. ______________________

(2 pts.) 2. ______________________

(3 pts.) 3. ______________________

Varsity Meet 3 – January 7, 2015
Round 5: Analytic Geometry

All answers must be in simplest exact form in the answer section
NO CALCULATOR ALLOWED

1. Write the equation of the circle with center $(-1, 9)$ and containing the point $(2, 5)$. Write your answer in the form $(x - A)^2 + (y - B)^2 = C$ (where A, B, and C are real numbers).

2. Find the x–intercept of the line tangent to the graph of $(x - 3)^2 + (y - 2)^2 = 25$ at the point $(6, 6)$.

3. The segment with endpoints at $P(4, 0)$ and $Q(0, 0)$ is the hypotenuse of a right triangle ΔPQR. The equation whose solution set gives all possible locations of the vertex R (except when $x = 0$ or $x = 4$) can be written in the form $Ax^2 + By^2 + Cx + Dy + E = 0$ for real numbers A, B, C, D, and E, where A and B are mutually prime positive integers. Find the value of $A + B + C + D + E$.

ANSWERS

(1 pt.) 1. ________________________

(2 pts.) 2. ________________________

(3 pts.) 3. ________________________

WORCESTER COUNTY MATHEMATICS LEAGUE

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Team Round

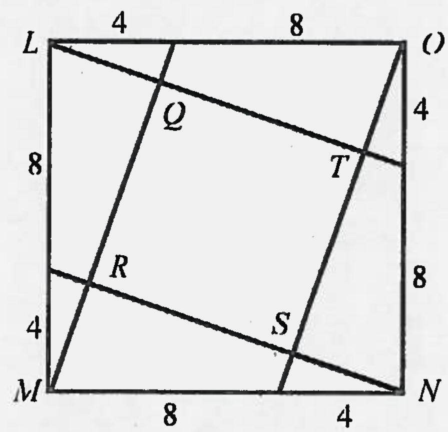
All answers must be in simplest exact form, and be written on the separate team answer sheet.

1. An ellipse is given by the equation $\frac{x^2}{13}+\frac{y^2}{4}=1$. Circle C contains both foci of the ellipse, and is tangent to the ellipse at one point below the x–axis. The line l is tangent to the ellipse at the ellipse's positive y–intercept. Find the point in the first quadrant where l and C intersect.

2. For how many positive six–digit integers is the product of the integer's digits equal to 12?

3. For any positive integer n, let $f(n)$ denote the number of positive factors of n. If m is a power of 2, $f(m)=a$, and $f(2^{18}\cdot m)=\frac{5}{3}a$, find the value of a.

4. In the figure, $LMNO$ is a square. Find the area of quadrilateral $QRST$.



5. If $f(x)=\sqrt[3]{g(x)+3}$ and $g^{-1}(x)=\frac{1}{2}x+4$, find the value of $f^{-1}(7)$.

6. Find all ordered pairs (m,n) such that m and n are both perfect squares, and $m-n=328$.

7. A "decimal" number in base 5 is more accurately called a "quinary" number. Write the repeating quinary number $0.030303\ldots_5$ as a base-ten decimal.

8. For how many integer values of x can x, $4x$, and 64 be the lengths of the three sides of a triangle?

9. Simplify:

$$5^{\sqrt{6}+1}\cdot 25^{-\sqrt{\frac{3}{2}}}$$

WORCESTER COUNTY MATHEMATICS LEAGUE

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Answers

Round 1: Similarity and Pythagorean Theorem

1. 16
2. $\frac{60}{13}$ or $4\frac{8}{13}$
3. 14

Shepherd Hill, Hudson, Auburn

Round 2: Algebra 1

1. 2286
2. $8\frac{8}{9}$ or $\frac{80}{9}$ or $8.\bar{8}$
3. 54

Tantasqua, Quaboag, QSC

Round 3: Functions

1. 26
2. -19
3. 243 → challenge: Answer is:

Assabet Valley, Westborough, QSC

Round 4: Combinatorics

1. 720
2. 192 → challenge: not overturned
3. 1320

Algonquin, Quaboag, St. John's

Round 5: Analytic Geometry

1. $(x+1)^2+(y-9)^2=25$
   or
   $(x-(-1))^2+(y-9)^2=25$
2. 14
3. -2

Douglas, St. John's, Doherty

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ANSWERS: Team Round

1. $(\sqrt{10}, 2)$

2. 120

3. 27

4. 57.6 or $57\frac{3}{5}$ or $\frac{288}{5}$

5. 174

6. $(6889, 6561)$ and $(1849, 1521)$

7. 0.125 or $\frac{1}{8}$

8. 9

9. 5

QSC, QSC, QSC, Marlborough, QSC, QSC, QSC, QSC, Clinton

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Team Round Answer Sheet

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM, AND BE WRITTEN ON THIS TEAM ANSWER SHEET.

(2 points each)

1. ______________________________

2. ______________________________

3. ______________________________

4. ______________________________

5. ______________________________

6. ______________________________

7. ______________________________

8. ______________________________

9. ______________________________

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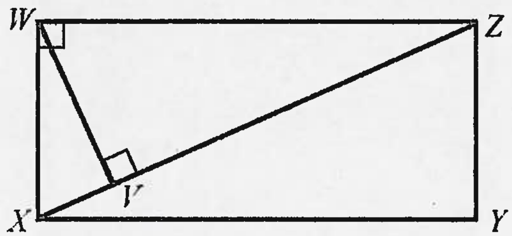
Solutions

Round 1: Similarity and Pythagorean Theorem

1. Since $\overline{AB}$ and $\overline{DE}$ are corresponding sides of similar triangles, the ratio between the triangles' side lengths is $\frac{AB}{DE} = \frac{9}{3} = \frac{3}{1}$. Then $\frac{3}{1} = \frac{AC}{DC} = \frac{24}{DC}$, meaning $DC = 8$. Lastly, $AD + DC = AC$, which gives $AD + 8 = 24$, so $AD = \mathbf{16}$.

2. As shown in the figure, the shortest distance from point W to the diagonal $\overline{XZ}$ will be the length of the segment that meets $\overline{XZ}$ at a right angle. Since $WX = 5$ and $WZ = 12$, $XZ = \sqrt{12^2 + 5^2} = \sqrt{169} = 13$.



Since ΔWXZ and ΔVWZ share $\angle WZX$ and both have a right angle, they are similar triangles. This means $\frac{WV}{WX} = \frac{WZ}{XZ}$, i.e. $\frac{WV}{5} = \frac{12}{13}$, which gives $WV = \mathbf{\frac{60}{13}}$.

3 Let $x = CD$, $y = BD$, and $z = DF$. $\Delta ABF \sim \Delta CDF$, meaning $\frac{AB}{BF} = \frac{CD}{DF}$, i.e.

$$\frac{50}{y+z} = \frac{x}{z} \quad \Longrightarrow \quad 50z = x(y+z)$$

Similarly, $\Delta EFB \sim \Delta CDB$, meaning

$$\frac{19\frac{4}{9}}{y+z} = \frac{x}{y} \quad \Longrightarrow \quad 19\frac{4}{9}y = x(y+z)$$

This implies $50z = 19\frac{4}{9}y$, so

$$y = \frac{50}{19\frac{4}{9}}z = \frac{50}{\left(\frac{175}{9}\right)}z = \frac{450}{175}z = \frac{18}{7}z$$

Then $50z = x\left(\frac{18}{7}z + z\right) = x\left(\frac{25}{7}z\right)$, which gives $50 = \frac{25}{7}x$, so $x = \frac{7}{25} \cdot 50 = \mathbf{14}$

Round 2: Algebra 1

1. Solution 1: Since $6 + 5 = 11$, the $6:5$ ratio indicates that $\frac{6}{11}$ of the total votes were cast for Ann. Then Ann received $\frac{6}{11} \times 4191 = \mathbf{2286}$ votes.

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Solution 2: If Ann received x votes and Barney received y votes, we have $x + y = 4191$ and $\frac{x}{y} = \frac{6}{5}$. The second equation gives $5x = 6y$, meaning $y = \frac{5}{6}x$. Substituting this into the other equation gives $x + \frac{5}{6}x = \frac{11}{6}x = 4191$, so that $x = \frac{6}{11} \times 4191 = \mathbf{2286}$.

2. If x gallons are drained, then $(20 - x)$ gallons of 30% solution remain, and x gallons of 75% solution must be added. This gives the equation:

$$0.30(20 - x) + 0.75x = 0.50 \times 20$$
$$0.30 \times 20 - 0.30x + 0.75x = 10$$
$$6 + 0.45x = 10$$
$$0.45x = 4$$
$$x = \frac{4}{0.45} = \frac{4}{\left(\frac{45}{100}\right)} = \frac{400}{45} = \mathbf{\frac{80}{9}}$$

3. Solution 1: We apply the formula $rt = d$ (where r is rate of speed, t is time, and d is distance) to the first part, last part, and entire commute, as indicated in the table.

	First part	Last part	Entire Commute
rate of speed r	20	x	$32\frac{1}{7}$
time t	t_1	t_2	$t_1 + t_2$
distance d	$\frac{2}{5}d$	$\frac{3}{5}d$	d
$rt = d$	$20t_1 = \frac{2}{5}d$	$xt_2 = \frac{3}{5}d$	$32\frac{1}{7}(t_1 + t_2) = d$

The three equations in the final row provide a system, which can be solved for x. Solving the first equation for d, we have:

$$d = \frac{5}{2} \times 20t_1 = 50t_1$$

Substituting this into the second equation gives

$$xt_2 = \frac{3}{5}(50t_1) = 30t_1$$

which implies

$$x = 30\frac{t_1}{t_2}$$

Substituting $d = 50t_1$ into the third equation gives

$$32\frac{1}{7}(t_1 + t_2) = 50t_1$$

$$32\frac{1}{7}t_1 + 32\frac{1}{7}t_2 = 50t_1$$

$$32\frac{1}{7}t_2 = 17\frac{6}{7}t_1$$

$$\frac{t_1}{t_2} = \frac{32\frac{1}{7}}{17\frac{6}{7}} = \frac{\left(\frac{225}{7}\right)}{\left(\frac{125}{7}\right)} = \frac{225}{125} = \frac{9}{5}$$

Plugging this value of $\frac{t_1}{t_2}$ into the previous equation, we see that

$$x = 30 \times \frac{9}{5} = \frac{270}{5} = \mathbf{54}$$

Solution 2: Assume the trip is 100 miles long. Then the first $\frac{2}{5} \cdot 100 = 40$ miles of the trip were traveled at an average of 20 mph, meaning this portion of the trip took $\frac{40 \text{ miles}}{20 \text{ mph}} = 2$ hours. Since the entire trip was traveled at an average of $32\frac{1}{7}$ mph, the whole trip took

$$\frac{100 \text{ miles}}{32\frac{1}{7} \text{ mph}} = \frac{100 \text{ miles}}{\frac{225}{7} \text{ mph}} = \frac{700 \text{ miles}}{225 \text{ mph}} = \frac{28}{9} \text{ hours}$$

Then the last $\frac{3}{5} \cdot 100 = 60$ miles of the trip took $\frac{28}{9} - 2 = \frac{28}{9} - \frac{18}{9} = \frac{10}{9}$ hours. The average speed is therefore

$$\frac{60 \text{ miles}}{\frac{10}{9} \text{ hours}} = \frac{540 \text{ miles}}{10 \text{ hours}} = \mathbf{54} \text{ mph}$$

Round 3: Functions

1. $f(-3) = (-3)^2 - (-3) - 1 = 9 + 3 - 1 = 11$
Therefore $g(f(-3)) = g(11) = 3 \times 11 - 7 = 33 - 7 = \mathbf{26}$

2. $f(g(x)) = g(x) + 4 = -2x - 5$, meaning $g(x) = -2x - 5 - 4 = -2x - 9$.
Then $g(f(x)) = -2(f(x)) - 9 = -2(x + 4) - 9 = -2x - 8 - 9 = -2x - 17$.
Hence $a = -2$, $b = -17$, and $a + b = \mathbf{-19}$.

3. Solution 1: $f^{-1}(8) = 81$ implies that $f(81) = 8$.

Since $f(ab) = f(a) + f(b)$, $8 = f(81) = f(9 \times 9) = f(9) + f(9) = 2 \cdot f(9)$, meaning $f(9) = 8 \div 2 = 4$.

Similarly, $4 = f(9) = f(3 \times 3) = f(3) + f(3) = 2 \cdot f(3)$, so $f(3) = 4 \div 2 = 2$.

Then we have $f(81) + f(3) = 8 + 2 = 10$, i.e. $f(81 \times 3) = f(243) = 10$. Applying f^{-1} to both sides of this equation, we have $f^{-1}(10) = f^{-1}(f(243)) = \mathbf{243}$.

Solution 2: Apply the rule $f(ab) = f(a) + f(b)$ to $a = f^{-1}(x)$ and $b = f^{-1}(y)$:

$$f(f^{-1}(x) \cdot f^{-1}(y)) = f(f^{-1}(x)) + f(f^{-1}(y)) = x + y$$

Applying f^{-1} to both sides, we have:

$$f^{-1}\left(f(f^{-1}(x) \cdot f^{-1}(y))\right) = f^{-1}(x) \cdot f^{-1}(y) = f^{-1}(x + y)$$

Then $81 = f^{-1}(8) = f^{-1}(4 + 4) = f^{-1}(4) \cdot f^{-1}(4) = (f^{-1}(4))^2$, which implies that $f^{-1}(4) = \sqrt{81} = 9$. (We know it cannot be -9, since the domain of f was given as $(0, \infty)$).

Similarly, $9 = f^{-1}(4) = f^{-1}(2 + 2) = f^{-1}(2) \cdot f^{-1}(2) = (f^{-1}(2))^2$, meaning $f^{-1}(2) = \sqrt{9} = 3$.

Then $f^{-1}(10) = f^{-1}(8 + 2) = f^{-1}(8) \cdot f^{-1}(2) = 81 \cdot 3 = \mathbf{243}$.

Round 4: Combinatorics

1. To fill 3 distinguishable positions from a pool of 10 people, there will be

$${}_{10}P_3 = \frac{10!}{(10-3)!} = \frac{10!}{7!} = 10 \cdot 9 \cdot 8 = \mathbf{720} \text{ possibilities.}$$

2. We separately count those shoes with both straps and bows, those with just straps, those with just bows, and those with neither.

Straps and bows: $4 \cdot 5 \cdot 7 = 140$ possibilities
Bows only: $4 \cdot 7 = 28$ possibilities
Straps only: $4 \cdot 5 = 20$ possibilities
Neither: 4 possibilities

Altogether, there are $140 + 28 + 20 + 4 = \mathbf{192}$ possible styles.

3. We separately count the words with no N, those with one N, and those with two Ns.

No N: In this case there are five letters remaining, and so there are

$_5P_5 = \frac{5!}{(5-5)!} = \frac{5!}{1} = 120$ words possible.

One N: In this case, the one N can occupy $_5C_1 = 5$ different positions within a word. Once the N is placed, the five non-N letters have four possible positions, meaning $_5P_4 = \frac{5!}{(5-1)!} = 120$ possible arrangements. Then altogether, this case gives $5 \cdot 120 = 600$ words.

Two Ns: The two Ns can occupy $_5C_2 = \frac{5!}{(5-2)!\cdot 2!} = \frac{120}{12} = 10$ positions within a word. Then, the remaining letters have three open positions, so $_5P_3 = \frac{5!}{(5-3)!} = \frac{120}{2} = 60$ possible arrangements, giving $10 \cdot 60 = 600$ words in this case.

Summing the three cases, there are $120 + 600 + 600 = \mathbf{1320}$ total words.

Round 5: Analytic Geometry

1. Since the center is $(-1, 9)$, the equation is of the form $(x+1)^2 + (y-9)^2 = r^2$. Since the circle contains $(2, 5)$, we have

$$(2+1)^2 + (5-9)^2 = r^2$$
$$3^2 + (-4)^2 = r^2$$
$$25 = r^2$$

Then the equation is $\mathbf{(x+1)^2 + (y-9)^2 = 25}$

2. The center of the given circle is at $(3, 2)$, and the slope of the segment with endpoints at $(3, 2)$ and $(6, 6)$ is

$$m = \frac{6-2}{6-3} = \frac{4}{3}$$

Since the line tangent to a circle at a point is perpendicular to the radius at that point, the slope of the line tangent to the circle at $(6, 6)$ is $-\frac{1}{m} = -\frac{3}{4}$. Using point-slope form, the equation of this line is

$$y - 6 = -\frac{3}{4}(x - 6)$$

To find the x-intercept, set y equal to 0:

$$0 - 6 = -\frac{3}{4}(x - 6)$$
$$(-6) \cdot \left(-\frac{4}{3}\right) = 8 = x - 6$$
$$x = \mathbf{14}$$

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3. Because $\overline{PQ}$ is the hypotenuse of ΔPQR, we have $PR^2 + QR^2 = PQ^2$. If R is located at the point (x, y), this gives

$$\left(\sqrt{(x-4)^2 + (y-0)^2}\right)^2 + \left(\sqrt{(x-0)^2 + (y-0)^2}\right)^2 = 4^2$$
$$(x^2 - 8x + 16 + y^2) + (x^2 + y^2) = 16$$
$$2x^2 + 2y^2 - 8x = 0$$
$$x^2 + y^2 - 4x = 0$$

Then $A + B + C + D + E = 1 + 1 + (-4) = \mathbf{-2}$

Team Round

1. The equation of the ellipse is in the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$, so the focus is

$$f = \sqrt{a^2 - b^2} = \sqrt{13 - 4} = \sqrt{9} = 3$$

Then C contains the two foci $(3, 0)$ and $(-3, 0)$. This means the circle is symmetric about the y–axis. Since the ellipse is too, if they have only one intersection point below the x-axis, this point must be on the y–axis. $b = \sqrt{4} = 2$, meaning this point is $(0, -2)$.

Because the circle's center must lie on the y–axis, its equation is $x^2 + (y - k)^2 = r^2$ for some k and r. Plugging in the points $(3, 0)$ and $(0, -2)$, we have:

$$3^2 + (0 - k)^2 = r^2 \qquad\qquad 0^2 + (-2 - k)^2 = r^2$$
$$9 + k^2 = r^2 \qquad\qquad k^2 + 4k + 4 = r^2$$

Subtracting the two equations, we have $5 - 4k = 0$, i.e. $k = \frac{5}{4}$. Then

$$r^2 = 9 + \left(\frac{5}{4}\right)^2 = 9 + \frac{25}{16} = \frac{144}{16} + \frac{25}{16} = \frac{169}{16}$$

This means the equation of the circle is $x^2 + \left(y - \frac{5}{4}\right)^2 = \frac{169}{16}$. The line l is horizontal and contains the point $(0, 2)$, so its equation is $y = 2$. Substituting 2 for y in the circle's equation, we have:

$$x^2 + \left(2 - \frac{5}{4}\right)^2 = \frac{169}{16}$$
$$x^2 + \left(\frac{3}{4}\right)^2 = \frac{169}{16}$$
$$x^2 = \frac{169}{16} - \frac{9}{16} = \frac{160}{16} = 10$$
$$x = \pm\sqrt{10}$$

So the point of intersection on the first quadrant is $(\sqrt{\mathbf{10}}, \mathbf{2})$.

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2. There are three ways of factoring 12 into 6 integers between 1 and 9 (inclusive):

$$12 = 6 \cdot 2 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \quad \text{(Case 1)}$$
$$12 = 4 \cdot 3 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \quad \text{(Case 2)}$$
$$12 = 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1 \cdot 1 \quad \text{(Case 3)}$$

Each case above gives multiple six-digit numbers, by reordering the digits. In Case 1, since there are 6 total numbers, but 4 (identical) '1's, the we have

$$\frac{6!}{4!} = 6 \cdot 5 = 30$$

six-digit numbers. Case 2 is analogous, and will also give 30 numbers. In Case 3, there are three '1's and two '2's, so we must divide by $3! \cdot 2!$ to get

$$\frac{6!}{3! \cdot 2!} = \frac{720}{6 \cdot 2} = \frac{720}{12} = 60$$

Combing all three cases, there are $30 + 30 + 60 = \mathbf{120}$ such numbers.

3. Solution 1: If $m = 2^n$, then m has $n + 1$ factors (namely $2^0, 2^1, \ldots, 2^n$). This means $a = n + 1$.

Since $f(2^{18} \cdot m) = \frac{5}{3}a$,

$$\frac{5}{3}a = (n + 18 + 1)$$

$$\text{which gives } a = \frac{3}{5}(n + 19)$$

By substitution,

$$n + 1 = \frac{3}{5}(n + 19)$$
$$n + 1 = \frac{3}{5}n + \frac{57}{5}$$
$$\frac{2}{5}n = \frac{52}{5}$$
$$n = 26$$

Then $a = n + 1 = \mathbf{27}$.

Solution 2: Let $m = 2^x$. Then $f(2^x) = a = x + 1$, and $m = 2^{a-1}$.

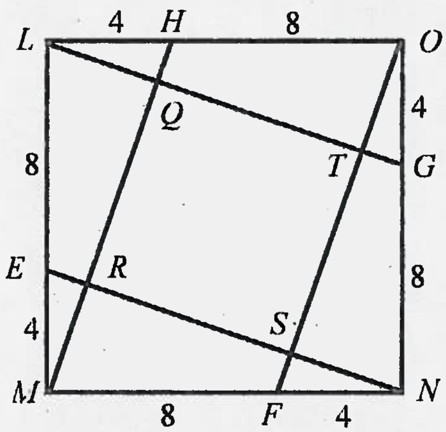
This means $\frac{5}{3}a = f(2^{18} \cdot m) = f(2^{18} \cdot 2^{a-1}) = f(2^{a+17}) = a + 17 + 1$. This gives:

$$\frac{5}{3}a = a + 18$$

$$\frac{2}{3}a = 18$$

$$a = 18 \cdot \frac{3}{2} = \mathbf{27}$$

4. . Solution 1: First, note that quadrilateral $QRST$ is a square. One way to see this is that the measured lengths along the perimeter of $LMNO$ are invariant under a 90° rotation (that is, the figure is the same before and after such a rotation); therefore the lines drawn inside square $LMNO$, which form $QRST$, must also be invariant under a 90° rotation. For this to be possible of $QRST$, it must be a square.



As this ensures there are four right angles at R, we have that $\Delta EMN \sim \Delta MRN \sim \Delta ERM$. By the Pythagorean Theorem,

$$EN = \sqrt{4^2 + 12^2} = \sqrt{160} = 4\sqrt{10}$$

By similarity, $\frac{MR}{ER} = \frac{RN}{RM} = \frac{MN}{EM} = \frac{12}{4} = 3$. If $RN = x$, then $RM = \frac{x}{3}$ and $ER = \frac{x}{9}$.

Then $4\sqrt{10} = EN = ER + RN = \frac{x}{9} + x = \frac{10}{9}x$, meaning $x = \frac{36\sqrt{10}}{10}$. This implies the area of ΔMRN is

$$\frac{1}{2}bh = \frac{1}{2} \cdot RN \cdot RM = \frac{1}{2} \cdot x \cdot \frac{x}{3} = \frac{1}{2} \cdot \frac{36\sqrt{10}}{10} \cdot \frac{36\sqrt{10}}{30}$$

$$= \frac{1}{2} \cdot \frac{18\sqrt{10}}{5} \cdot \frac{6\sqrt{10}}{5} = \frac{108 \cdot 10}{50} = \frac{108}{5}$$

Square $LMNO$ comprises four congruent triangles, each of area $\frac{108}{5}$, in addition to square $QRST$. Therefore the area of $QRST$ is

$$12^2 - 4 \cdot \frac{108}{5} = 144 - \frac{432}{5} = 144 - 86.4 = \mathbf{57.6}$$

Solution 2: Follow Solution 1 to find $RN = x = \frac{36\sqrt{10}}{10}$. Then we know $SN = \frac{x}{3}$, meaning $RS = \frac{2}{3}x = \frac{72\sqrt{10}}{30} = \frac{12\sqrt{10}}{5}$. Therefore the area of $QRST$ is

$$\left(\frac{12\sqrt{10}}{5}\right)^2 = \frac{144 \cdot 10}{25} = \frac{144 \cdot 10 \cdot 4}{100} = \frac{5760}{100} = \mathbf{57.6}$$

5. Solution 1: Since $g^{-1}(x) = \frac{1}{2}x + 4$,

$$x = g^{-1}(g(x)) = \frac{1}{2}g(x) + 4$$

$$g(x) = (x-4)\cdot 2 = 2x - 8$$

Then $f(x) = \sqrt[3]{g(x)+3} = \sqrt[3]{2x-8+3} = \sqrt[3]{2x-5}$. This means

$$x = f\left(f^{-1}(x)\right) = \sqrt[3]{2f^{-1}(x) - 5}$$
$$x^3 = 2f^{-1}(x) - 5$$
$$f^{-1}(x) = \frac{x^3+5}{2}$$

Therefore $f^{-1}(7) = (7^3+5) \div 2 = (343+5) \div 2 = 348 \div 2 = \mathbf{174}$.

Solution 2: If $f^{-1}(7) = x$, then $f(x) = \sqrt[3]{g(x)+3} = 7$. This means

$$g(x) = 7^3 - 3 = 343 - 3 = 340$$

This is equivalent to $x = g^{-1}(340) = \frac{1}{2}\cdot 340 + 4 = 170 + 4 = \mathbf{174}$.

6. Since $x^2 - y^2 = (x-y)(x+y)$, $328 = x^2 - y^2$ for integers x and y if and only if 328 can be factored as $328 = (x-y)(x+y)$.

The prime factorization of 328 is $328 = 2^3 \cdot 41$, which we use to determine that 328 can be written as the following four products of positive integers:

$1 \cdot 328$ $\quad$ $2 \cdot 164$ $\quad$ $4 \cdot 82$ $\quad$ $8 \cdot 41$

We then check which of these pairs can be written as $(x-y)\cdot(x+y)$. In each of the four cases, this requires solving the linear system (for appropriate values of a and b)

$$x - y = a$$
$$x + y = b$$

Adding the two equations, we have:

$$2x = a + b$$
$$x = \frac{a+b}{2}$$

Subtracting the two equations instead, we have:

$$-2y = a - b$$
$$y = \frac{a-b}{-2} = \frac{b-a}{2}$$

For x and y to be integers, $a+b$ (and $a-b$) must therefore be even, which is not the case when $a = 1$ and $b = 328$, or when $a = 8$ and $b = 41$.

This leaves $a = 2$, $b = 164$ and $a = 4$, $b = 82$. In the first case,

$$x = \frac{2+164}{2} = \frac{166}{2} = 83$$
$$y = \frac{164-2}{2} = \frac{162}{2} = 81$$

meaning $(m,n) = (83^2, 81^2) = (\mathbf{6889}, \mathbf{6561})$.

In the second case,

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$$x = \frac{4+82}{2} = \frac{86}{2} = 43$$

$$y = \frac{82-4}{2} = \frac{78}{2} = 39$$

meaning $(m,n) = (43^2, 39^2) = \mathbf{(1849, 1521)}$.

7. The value of $0.030303\ldots_5$ is $3 \cdot 5^{-2} + 3 \cdot 5^{-4} + 3 \cdot 5^{-6} + \cdots$, which is a geometric series whose first term is $a = \frac{3}{25}$ and whose common ratio is $r = \frac{1}{25}$. Its sum is therefore

$$\frac{a}{1-r} = \frac{\frac{3}{25}}{1-\frac{1}{25}} = \frac{\frac{3}{25}}{\frac{24}{25}} = \frac{3}{24} = \frac{1}{8}$$

In base ten, $\frac{1}{8} = \mathbf{0.125}$.

8. The sides of a triangle must satisfy the triangle inequality, so we need:

(i) $x + 4x > 64$
(ii) $x + 64 > 4x$
(iii) $4x + 64 > x$

The first inequality can be rewritten $5x > 64$, meaning $x > \frac{64}{5} = 12.8$. Since we require that x take integer values, this means $x \geq 13$.

The second inequality is equivalent to $64 > 3x$, meaning $x < \frac{64}{3} = 21\frac{1}{3}$. So $x \leq 21$.

The last inequality becomes $64 > -3x$, which gives $x > -\frac{64}{3}$. As x must clearly be positive, this case does not add any new restriction on x.

Therefore, x can take **9** integer values, from 13 to 21, inclusive.

9.

$$25^{-\sqrt{\frac{3}{2}}} = (5^2)^{-\sqrt{\frac{3}{2}}} = 5^{2\cdot\left(-\sqrt{\frac{3}{2}}\right)} = 5^{\sqrt{4}\cdot\left(-\sqrt{\frac{3}{2}}\right)} = 5^{\left(-\sqrt{\frac{4\cdot 3}{2}}\right)} = 5^{(-\sqrt{6})}$$

Therefore:

$$5^{\sqrt{6}+1} \cdot 25^{-\sqrt{\frac{3}{2}}} = 5^{\sqrt{6}+1} \cdot 5^{-\sqrt{6}} = 5^{\sqrt{6}+1-\sqrt{6}} = 5^1 = \mathbf{5}$$